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A wave-guide model for turbulent shear flow

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It is shown that for a dynamical system admitting wave propagation modes (i.e. a wave-guide) the cross-power spectral density for stationary random fluctuations in the system will be dominated by the waves if they are lightly damped, the reason being that these can correlate over large distances of the order the inverse of the damping ratio. For a turbulent shear flow the wave propagation constant is obtained approximately from the solution of the Orr–Sommerfeld problem for the mean flow. Numerical calculations for a flat-plate boundary layer produce results for the streamwise dependence of the cross-power spectral density for the surface pressure fluctuations in good qualitative and quantitative agreement with measurements. An exception is the convection velocity for which the theory predicts a value that is somewhat too low.

1. Introduction

The present work grew out of an attempt to relate the statistical properties of wall-pressure fluctuations beneath a turbulent boundary layer to the overall gross properties of the mean shear flow. The aim was to find a way to estimate the cross-spectral density from the one-point power spectrum and the mean velocity distribution. It was hoped that this statistical quantity, which is required, for example, in the calculation of the response of a flexible flight structure to boundary-layer noise, could be obtained with a considerable saving in the amount of measured data needed. This goal was partly reached. In addition, however, certain more general properties of shear flow turbulence were discovered.

A great deal of theoretical work has been devoted to investigations of the statistical properties of the wall-pressure fluctuations. Virtually all the published papers on this subject have been based on writing the differential equation for the pressure as a Poisson equation with non-linear fluctuating-stress terms and terms involving interaction with the mean shear considered as 'source terms'. This approach has been tried by, among others, Kraichnan (1956) and Lilley & Hodgson (1960). The difficulty with this is that the theory cannot be carried very far without making several assumptions about the form of the source terms.

The present approach is completely different in that an attempt is made to relate the fluctuating pressures to the stability problem for the mean flow. It has been pointed out by Mollo-Christensen (1967) that measurements of pressure correlations in jets and free shear layers give strong evidence of a wave-like character of the fluctuations, despite the fact that they are completely random.

The importance of the stability aspects of turbulent shear flow has of course been recognized for some time. The stability of the mean flow plays a fundamental role in Malkus's (1956) well-known theory of turbulent shear flow. He hypothesized that, among the turbulent fluctuations, there were at least some that were marginally stable and that they must be selected within a class that gives maximum viscous dissipation. The theory was successful in predicting a mean velocity profile in good agreement with experiment, but it now appears (Reynolds & Tiederman 1967) that his success might have been fortuitous. It will later be seen that the present theory allows one to investigate Malkus's hypotheses regarding the stability of the mean flow.

2. Random disturbances in a wave-guide

In order to bring out the importance of wave modes for random disturbances and their proper interpretation, consider for simplicity the case of a one-dimensional wave-guide into which stationary random disturbances are introduced. Let the random signal be of the form $p(x,t)$ and assume further homogeneity in x so that the correlation function

$$R_{pp} = \overline{p(x,t)\,p(x+\xi,t+\tau)}, \tag{1}$$

is independent of x and t (bar denotes ensemble average). In this problem it will be more instructive to work instead with the cross-spectral density S_{pp} here defined as

$$S_{pp}(\xi;\,\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau} R_{pp}(\xi,\tau)\,d\tau. \tag{2}$$

Let $\hat{p}(k,\omega)$ denote the generalized Fourier transform of $p(x,t)$ with respect to x and t,

$$\hat{p} = \iint_{-\infty}^{\infty} e^{-i(kx-\omega t)}\,p(x,t)\,dx\,dt.$$

Then the wave-number-frequency spectrum $\hat{S}_{pp}(k,\omega)$ being defined as the Fourier transform of S_{pp} with respect to ξ, is related to $\hat{p}$ through the formula

$$\hat{S}_{pp}(k,\omega) = \frac{\overline{\hat{p}(k,\omega)\,\hat{p}^*(k',\omega')}}{\delta(k-k',\omega-\omega')}, \tag{3}$$

where δ is the generalized Dirac delta function and star denotes complex conjugate. Thus,

$$S_{pp}(\xi,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}\,\hat{S}_{pp}(k,\omega)\,dk. \tag{4}$$

The random disturbances will excite the wave modes in the wave-guide and the Fourier transform of p will thus contain terms of the form

$$\frac{C^{(n)}}{k-\alpha^{(n)}}, \tag{5}$$

where $\alpha^{(n)} = \alpha_R^{(n)} + i\alpha_I^{(n)}$ are the wave propagation constants. Here we have given the result as it would appear if one first carried out the Fourier transform with respect to t and then with respect to x. If the transform with respect to x were carried out first, the result would be of the form $C/(\omega-\omega^{(n)})$, where

$$\omega^{(n)} = \omega_R^{(n)} + i\omega_I^{(n)}$$

are the complex eigenfrequencies for real wave-numbers. The final result for the cross-spectral density obtained after inversion of the x-transform would, of course, be the same in both cases (cf. analysis following (26) below) corresponding physically to the correlation of wave-like disturbances each varying like $\exp(-\alpha^{(n)}(\omega)\,x)$ downstream from their origin.

The coefficients $C^{(n)}$ will generally be functions of k and ω. Note that in order for the disturbance field to be statistically homogeneous in x, the waves must all be damped, otherwise the effects of the ends of the wave-guide will always be felt, and the statistical properties will thus depend on x.† Hence $\alpha_I^{(n)}$ must be positive for waves travelling in the positive x-direction (i.e. for $c_R^{(n)} = (\omega/\alpha^{(n)})_R > 0$) and negative for waves travelling in the negative x-direction ($(\omega/\alpha^{(n)})_R < 0$).

Substituting (5) into (4) we can evaluate the contribution from the wave modes by contour integration. Considering for simplicity only waves travelling in the positive x-direction ($c_R^{(n)} > 0$) we obtain

$$S_{pp}(\xi,\omega) = i\sum_n\sum_m \frac{\Phi^{(n,m)}(\omega)}{\alpha^{(n)}-\alpha^{(m)*}}\,e^{i\alpha^{(n)}\xi}, \quad \text{for} \quad \xi > 0, \tag{6}$$

$$S_{pp}(-\xi,\omega) = S_{pp}^*(\xi,\omega),$$

where

$$\Phi^{(n,m)}(\omega) = \frac{\overline{G^{(n)}(\alpha^{(n)},\omega)\,G^{(m)*}(k',\omega')}}{\delta(\alpha^{(n)}-k',\omega-\omega')},$$

and the contributions from the poles of $C^{(n)}$, $C^{(m)}$, if any, have been omitted. Of special interest is the case when one or more of the wave modes are lightly damped so that $|\alpha_I^{(n)}| \ll |\alpha_R^{(n)}|$. Then the main contribution will arise from $m = n$ and be of the form

$$S_{pp}(\xi,\omega) = \sum_n \frac{1}{2\alpha_I^{(n)}}\,\Phi^{(n,n)}(\omega)\exp\left(i\alpha_R^{(n)}\xi - \alpha_I^{(n)}|\xi|\right). \tag{7}$$

Thus, for waves of equal order of amplitude, the least attenuated wave denoted by superscript zero, say, will generally give the largest contribution to the cross-spectral density. The interpretation of this result is that a wave can correlate significantly with other waves of the same frequency and phase that originate within a distance of order $(\alpha_I^{(n)})^{-1}$. Hence the waves will dominate in the cross-spectral density if lightly damped, because they will then generally correlate over a much larger distance than non-wave-like disturbances.

3. Random disturbances in a parallel shear flow

The result of the previous section, that the least damped travelling waves will dominate statistically for the homogeneous and stationary case, is of course true regardless of what particular physical wave propagation mechanism is con-

† An infinitely long wave-guide with linear response would of course be impossible in the unstable case.

sidered. In the present case we are concerned with a viscous shear flow for which it is well known from the theory of hydrodynamic stability that propagation of wave-like disturbances is possible.

For simplicity, we will consider the fluid to be incompressible. We split the random velocity field into a mean part and a fluctuating part by setting

$$U_i(x_i, t) = \overline{u}_i(x_i) + u_i(x_i, t). \tag{8}$$

The assumption will be made that the fluctuations are statistically stationary and homogeneous in the $x_1 = x$- and $x_3 = z$-directions and that the mean flow is parallel, i.e. that

$$\overline{u}_1 = U(x_2); \quad \overline{u}_2 = 0, \quad \overline{u}_3 = 0, \tag{9}$$

in accordance with the usual practice in hydrodynamic stability theory. The parallel-flow assumption is consistent with statistical homogeneity in the x_1-direction. It holds strictly only for a flow between parallel, infinite planes, but provides an excellent approximation in the case of the flow in an attached boundary layer. For a free shear layer or a jet it is not so good, but as will be discussed later, the assumption of statistical homogeneity in the x_1-direction will then have to be abandoned, anyway.

Substitution of (8) into the Navier–Stokes equations and subtraction of the mean part gives

$$\frac{\partial u_i}{\partial t} + \overline{u}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + T_i, \tag{10}$$

where
$$T_i = (\partial/\partial x_j)(\overline{u_j u_i} - u_j u_i). \tag{11}$$

By taking $\partial/\partial x_i$ of this and applying the continuity equation

$$\partial u_i/\partial x_i = 0 \tag{12}$$

we obtain
$$\frac{\partial^2 p}{\partial x_i^2} = -2\rho \frac{\partial u_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} + \rho \frac{\partial T_i}{\partial x_i}, \tag{13}$$

or, with $x_1 = x$, $x_2 = y$, $x_3 = z$,

$$\nabla^2 p = -2\rho U'(y)\frac{\partial v}{\partial x} + \rho \nabla . \mathbf{T}, \tag{14}$$

where $\mathbf{T} = (T_x, T_y, T_z)$ is the vector corresponding to (11). This equation, which looks deceptively simple, has attracted a great deal of attention. As originally suggested by Lighthill (1952) and Townsend (1956) one can solve (14) formally by assuming the right-hand side to be known and treat it as a Poisson equation. This approach has been followed by, among others, Kraichnan (1956) and Lilley & Hodgson (1960). Unfortunately, in order to obtain any quantitative results, one is forced to make several assumptions about the unknown right-hand side. Furthermore, because the linear terms $\nabla^2 p$ on the one hand and $2\rho U' \partial v/\partial x$ on the other are not treated in an equal manner, important aspects of the mathematical structure of the problem remain hidden, in particular the possibility of the occurrence of eigensolutions.

In order to produce an equation which contains all the linear terms on the left-hand side, we may use the second of (10), viz.

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v+T_y. \tag{15}$$

With this, $\partial v/\partial x$ can be eliminated from (14) and one can hence get an equation in terms of p, or alternatively one can eliminate $\nabla^2 p$ and thus produce an equation for v. The first course was followed in a preliminary paper (Landahl 1965). Here we will instead follow the second approach so that certain results from hydrodynamic stability theory can be more readily applied. Thus, by taking $\partial/\partial y$ of (14) and ∇^2 of (15) we obtain

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x}-\nu\nabla^4 v = q, \tag{16}$$

where

$$q = \nabla^2 T_y - \frac{\partial}{\partial y}(\nabla . \mathbf{T}). \tag{17}$$

If $\mathbf{T}$ and hence q are considered known, one can solve (16) subject to the homogeneous boundary conditions that v and $\partial v/\partial y$ vanish at the surface $y = 0$, and far away from it (for the boundary-layer case). The condition for $\partial v/\partial y$ at $y = 0$ follows from the application of the continuity equation (12). Having v, one can then proceed to calculate p, the quantity of primary interest, by solving (14). A more convenient equation for the pressure is obtained through the combination of (14) with the y-derivative of (15) which gives

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2} = \rho\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial v}{\partial y}-U'\frac{\partial v}{\partial x}-\nu\nabla^2\left(\frac{\partial v}{\partial y}\right)+\frac{\partial T_x}{\partial x}+\frac{\partial T_z}{\partial z}\right]. \tag{18}$$

From the results for p we may obtain the u- and w-components, if so desired, by integrating the remaining momentum equations.

We will now attempt to solve (16) by application of Fourier transformation in x, z and t. Letting

$$\hat{v} = \iiint_{-\infty}^{\infty} \exp\{-i(\mathbf{k}.\mathbf{r}-\omega t)\}\, v(\mathbf{r},t)\, d\mathbf{r}\, dt, \tag{19}$$

we obtain

$$\left.\begin{aligned} (Uk_x-\omega)\,\hat{\nabla}^2\hat{v} - U''k_x\hat{v}+i\nu\hat{\nabla}^4\hat{v} &= -i\hat{q}, \\ \hat{v}(0) = \hat{v}'(0) = \hat{v}(\infty) = \hat{v}'(\infty) &= 0, \end{aligned}\right\} \tag{20}$$

where $\hat{\nabla}^2 = d^2/dy^2 - k^2$, $\hat{q}$ is the triple Fourier transform of q and $\mathbf{k} = (k_x, k_z)$. To put this equation in a more familiar form we introduce non-dimensional quantities in the manner common in hydrodynamic stability problem (distances referred to boundary-layer thickness, δ; velocities referred to the free-stream velocity, U_∞, so that $U(y) = 1$ for $y \geqslant 1$). In the non-dimensional form of the equation, we also set

$$\left.\begin{aligned} k_x &= \alpha, \quad k_z = \beta, \\ \omega &= \alpha c, \quad \nu = 1/R, \quad \hat{v} = -ik\phi(y), \end{aligned}\right\} \tag{21}$$

where R is the Reynolds number based on the boundary-layer thickness and free-stream velocity. Then, (20) becomes

$$(U-c)(\phi''-k^2\phi)-U''\phi+\frac{i}{\alpha R}(\phi^{\text{iv}}-2k^2\phi''+k^4\phi) = \hat{q}/\alpha k, \tag{22}$$

which is recognized as a non-homogeneous Orr–Sommerfeld equation. A formal solution is most easily obtained through an expansion in terms of the eigenfunctions of the homogeneous problem (see, e.g. Eckhaus 1965). Let $c^{(n)}$ denote the (complex) eigenvalues of c for a given (real) set of α, β and R. Also, let $\phi^{(n)}$ be the associated eigenfunctions. Then, setting

$$\phi = \Sigma \hat{A}^{(n)}\phi^{(n)}(y), \tag{23}$$

one can show that

$$\hat{A}^{(n)} = \frac{1}{\alpha k(c-c^{(n)})}\int_0^\infty \hat{q}\tilde{\phi}^{(n)}\,dy, \tag{24}$$

where $\tilde{\phi}^{(n)}$ are the corresponding eigenfunctions for the adjoint problem

$$\left.\begin{aligned}(U-c)(\tilde{\phi}''-k^2\tilde{\phi})+2U'\tilde{\phi}'+(i/\alpha R)(\tilde{\phi}^{\text{iv}}-2k^2\tilde{\phi}''+k^4\tilde{\phi}) = 0,\\ \tilde{\phi}(0) = \tilde{\phi}'(0) = \tilde{\phi}(\infty) = \tilde{\phi}'(\infty) = 0,\end{aligned}\right\} \tag{25}$$

and the eigenfunctions have been normalized in such a manner that

$$\int_0^\infty \left(\frac{d\tilde{\phi}^{(n)}}{dy}\frac{d\phi^{(n)}}{dy}+k^2\tilde{\phi}^{(n)}\phi^{(n)}\right)dy = 1. \tag{26}$$

Let $k_x = \alpha_R^{(n)}$ be the (real) value of k_x for which $k_x c_R^{(n)} = \omega$. Then the denominator $k\alpha(c-c^{(n)}) = k(\omega-\alpha c^{(n)})$ in (24) will vary near the pole for a fixed (real) value of ω like (assuming that the variation of $c_I^{(n)}$ with ω is small)

$$\begin{aligned}-k(k_x-\alpha_R^{(n)})\frac{\partial}{\partial\alpha}(\alpha c^{(n)}) &\simeq -k\left[(k_x-\alpha_R^{(n)})\frac{\partial}{\partial\alpha}(\alpha c_R^{(n)})_{\alpha_R^{(n)}}+i\alpha_R^{(n)}c_I^{(n)}\right]\\ &\simeq -kc_g(k_x-\alpha_R^{(n)}-i\alpha_I^{(n)})\end{aligned}$$

using the usual approximate relationship between temporal and spatial amplification rate, c_g being the group velocity. It thus follows that (23) will have poles near the real axis of the form $(k_x-\alpha^{(n)})^{-1}$, where $\alpha^{(n)} = \alpha_R^{(n)}+i\alpha_I^{(n)}$ are the eigenvalues for the spatial case (complex α for given real values of ω, β and R).

Turning now to the pressure, we find through Fourier transformation of (18) that

$$\hat{p} = -\rho\frac{\alpha}{k}\left\{[(U-c)\phi'-U'\phi]+\frac{i}{\alpha R}(\phi'''-k^2\phi')+\frac{i}{\alpha}\left(\hat{T}_x+\frac{\beta}{\alpha}\hat{T}_z\right)\right\}. \tag{27}$$

Since $\hat{p}$ has terms that are linearly related to ϕ it follows that it, also, must have contributions of the form

$$\frac{C^{(n)}}{k_x-\alpha^{(n)}} \tag{28}$$

representing travelling waves. Applying the result (7) above (and extending it to propagation in two dimensions), we then find that the cross-spectral density must obtain its largest contribution from the least attenuated mode and be of the form

$$S_{pp}(\xi,\zeta,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Phi^{(0,0)}}{2\alpha_I^{(0)}}\exp(i\alpha_R^{(0)}\xi-\alpha_I^{(0)}|\xi|+ik_z\zeta)\,dk_z, \tag{29}$$

where $\Phi^{(0,0)}$ is a function of both k_z and ω, related to the spectral function of the coefficient $\hat{A}^{(0)}$ as obtained from the non-linear stress terms. The eigenvalue $\alpha^{(0)}$ is, of course, also a function of both ω and $k_z = \beta$, for a given R, and we cannot proceed any further at this stage without more information about its behaviour.

4. Eigenvalue calculations for a flat-plate boundary layer

We seek the lowest eigenvalue $\alpha^{(0)} = \alpha_R^{(0)} + i\alpha_I^{(0)}$ to the Orr–Sommerfeld problem

$$\left.\begin{aligned} (U-c)(\phi''-k^2\phi) - U''\phi + (i/\alpha R)(\phi^{\mathrm{iv}} - 2k^2\phi'' + k^4\phi) = 0,\\ \phi(0) = \phi'(0) = \phi(\infty) = \phi'(\infty) = 0, \end{aligned}\right\} \quad (30)$$

where $$k^2 = \alpha^2 + \beta^2$$

for given (real) values of $\omega = \alpha c$, R and β. This is the usual stability problem for a laminar boundary layer with the velocity distribution $U(y)$, except that now actual eigenvalues (for the spatial case) instead of just stability boundaries are required. (From the previous analysis it was concluded that the waves will always be found to be stable.) Since the Reynolds numbers of interest for the turbulent boundary layer are very high, it might seem appropriate to apply the same asymptotic methods that have been so successful in stability theory.

Unfortunately, the range of parameters of interest in the present problem is quite different from that encountered in the traditional stability problem. With the standard quantity used for reference values (boundary-layer thickness for length; free-stream speed for velocities) one finds that typical values are

$$|\alpha| = 10-100, \quad |\beta| = 0-1000,$$
$$R = 50{,}000-500{,}000.$$

Furthermore, because of the logarithmic part of the turbulent velocity profile, it turns out that U'', a quantity that is implicitly assumed to be of order unity in hydrodynamic stability theory, may be of the order of thousands near the wall. Lower figures can, of course, be arrived at by using some smaller reference length, for example, the displacement thickness δ^*, the viscous length $l^+ = \nu/u_\tau$, or the wall layer thickness (approximately $20l^+$), but then one needs to account for the fact that the edge of the boundary layer is located far out in terms of the reference length. It soon becomes clear that the traditional approach even with modifications to account for the large wave-numbers is not suitable, and that a direct numerical attack on the eigenvalue problem is the most profitable one.

In the numerical solution of (30) one is faced with a rather serious numerical stability problem. This is due to the small parameter $(\alpha R)^{-1}$ multiplying the highest derivative in the equation. An asymptotic analysis of the Orr–Sommerfeld equation shows that of the four linearly independent solutions, two solutions ϕ_1 and ϕ_2 vary moderately through the boundary layer and are closely approximated by the solutions of the second-order equation obtained from (30) by neglecting the viscous terms (the so-called inviscid Orr–Sommerfeld equation or Rayleigh equation). The remaining two solutions, ϕ_3 and ϕ_4, on the other hand, depend very strongly on R and grow or decay in an oscillatory manner extremely rapidly. A straightforward numerical integration will therefore, because of roundoff errors and other inaccuracies, eventually give a result dominated completely by the rapidly growing viscous solution, regardless of what starting conditions are used. Since the final result is likely to behave essentially like an inviscid solution, except near the wall and around the 'critical point' $U = c$, it is clear that such a straight-forward integration is bound to fail in practice. A method to cope with this difficulty was devised by Kaplan (1964). (For a description, see also Landahl & Kaplan 1965.) The method employs integration from the outer edge of the boundary layer and a filtering technique to remove excessive contaminations by the viscous solution at each integration step. Kaplan used this technique for the stability analysis of a laminar boundary layer over a flexible surface and, in view of his success, it was judged to be the best available method for the present problem.

A fairly extensive set of programs to handle the stability problem for a variety of velocity profiles and wall conditions (including those corresponding to compliant walls) was developed building-block fashion, around this method and put on the MIT IBM 7094 time sharing system.† This system allows direct communication with the computer, and turned out to be a rather essential tool for the present problem in that extensive numerical experimentation became feasible. For the integration of the differential equation a Runge–Kutta scheme was employed. Since a turbulent velocity profile varies rapidly in the region next to the wall, the program was designed to allow for a step size that could be varied in four different integration regions. For the velocity profile, Reichardt's (1951) expression for the wall region and logarithmic region, together with Coles's (1956) universal 'law of the wake', gives the formula

$$U/u_\tau = (1/\kappa)\ln(1+\kappa y^+) + \gamma(1 - e^{-y^+/\delta_v} - (y^+/\delta_v)\,e^{-0\cdot33y^+}) + 1\cdot38\,\{1+\sin[(2y-1)\,\tfrac{1}{2}\pi]\}, \qquad (31)$$

where
$$y^+ = yu_\tau/\nu$$

and $u_\tau = (\frac{1}{2}c_f)^{\frac{1}{2}}$ (c_f = friction coefficient) is the wall friction velocity normalized with the free-stream velocity. For the constants κ, γ, and δ_v the following values were used:

$$\kappa = 0\cdot4, \quad \gamma = 7\cdot4, \quad \delta_v = 11\cdot0.$$

The friction velocity u_τ was chosen so as to give the same displacement thickness Reynolds number as in the experiments by Willmarth & Wooldridge (1962). The velocity profile is shown in figure 1. The corresponding Reynolds number based on boundary-layer thickness for this case was approximately $4\cdot10^5$. The Reynolds number used in the Orr–Sommerfeld equation, however, was actually left as a free parameter in order that the accuracy and convergence of the Kaplan purification scheme for extreme values of the Reynolds number could be assessed. The calculation of an eigenvalue proceeds as follows. First, the integration is started at the edge of the boundary layer with starting conditions corresponding to the solution ϕ_3 using a guessed value for the eigenvalue. A second integration is carried out for ϕ_1 subject to the appropriate starting condition, but this time the filtering technique is employed to avoid contamination by ϕ_3. These two

† For a more detailed description of these calculations, see Landahl (1966). Further numerical results, including the eigensolutions, will be presented in a later publication.

solutions are then combined to satisfy the condition of zero tangential velocity at the wall. The condition of zero normal velocity serves as the eigenvalue criterion. If this is not satisfied to within a specified tolerance, a second trial value for the eigenvalue is used; from the third trial on, the program selects values using a Lagrangian interpolation formula to speed the convergence. Whenever

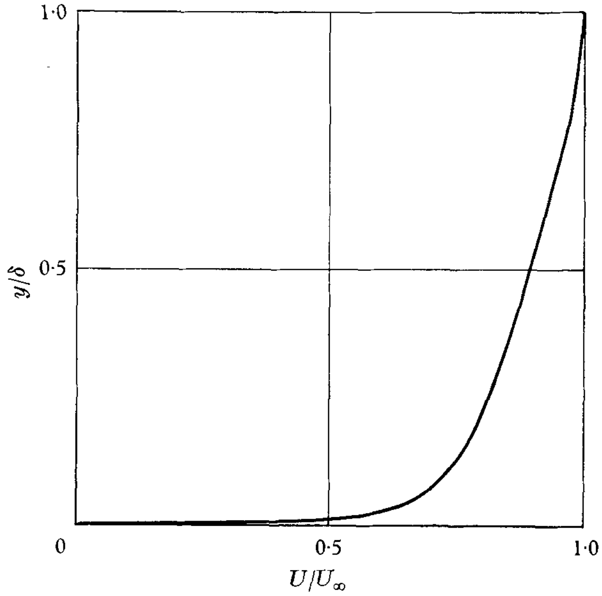

FIGURE 1. Mean velocity profile used. Edge of boundary layer at $y^+ = 12770$. Displacement thickness $\delta^*/\delta = 0{\cdot}1228$. $R = 402{,}300$, $R_{\delta^*} = 49{,}300$, $u_\tau = 0{\cdot}0315$.

the initial guess was reasonably close to the correct value, the program usually homed in on the eigenvalue in 4 tries or less. For the highest Reynolds number values employed, it was necessary to start the search very close to the eigenvalue; otherwise the calculation would not converge. The calculated results were subjected to the following checks.

(*a*) The number of integration steps was gradually increased until the eigenvalue no longer changed substantially.

(*b*) The integration routine was checked for the velocity distribution

$$U(y) = \text{const.} = 1,$$

for which an analytical solution can readily be derived.

(*c*) The eigenfunctions were calculated and their smoothness checked.

(*d*) The program was applied to the Blasius velocity profile and the results compared to those of earlier calculations.

From these checks it was determined that the filtering technique to supress numerical instability works for surprisingly large Reynolds numbers, but that its range of applicability seems to fall somewhat short of what is desired for a comparison with the experimental results of Willmarth & Wooldridge (1962). The highest value of R for which self-consistent results for the eigenvalues were obtained was $R = 40{,}000$ which is about one tenth of the laboratory value.

Although the program occasionally yielded results for some higher Reynolds number values, convergence difficulties were experienced, as well as increased sensitivity to changes in integration step size, etc., and these results were therefore judged less reliable. They indicate, however, that, somewhat surprisingly, there is still a very small but noticeable variation of the eigenvalues with the Reynolds number at $R = 40{,}000$.

Considerable numerical experimentation was carried out to establish the effect of integration step size. In order to make sure that the effects of the rapid

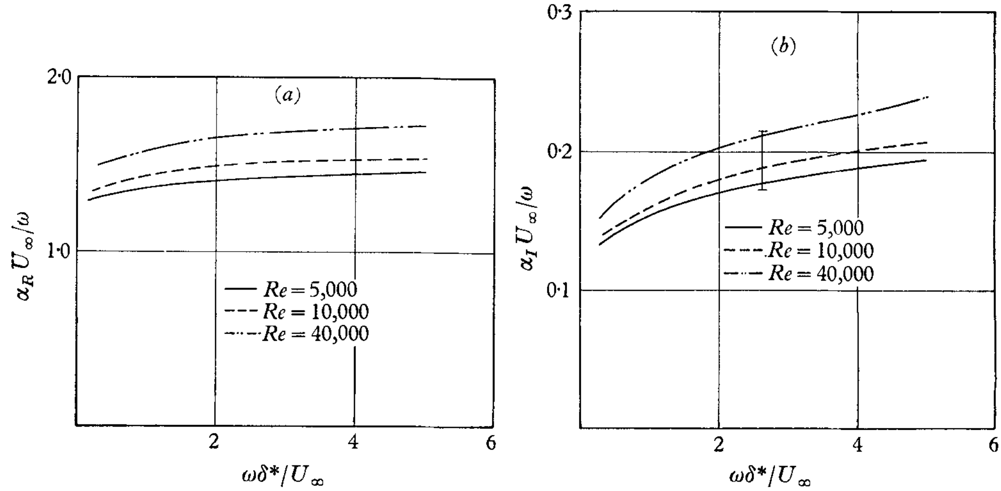

FIGURE 2. Propagation constants for zero wave angle. (*a*) Real part. (*b*) Imaginary part.

variation of the velocity profile in the viscous wall layer were properly accounted for, several calculations were carried out with a step size as small as $\Delta y = 0{\cdot}0001$ near the wall. Clearly, without the feature of the program that allows a variable step size to be used, such a calculation would be prohibitively time consuming. It was also discovered that in most cases one could dispense with the integration in a substantial outer portion of the boundary layer, since the disturbance was found to be very nearly irrotational (i.e. to vary as $\exp(-ky)$) for $ky > 3$. Hence, for the higher frequencies the range of integration could be reduced by a factor of 5 to 10. In most of the results presented below, 150 steps and a uniform step size were used, but a large portion was checked by application of a substantial variation of the number of steps and the step size distribution.

Figure 2 shows the results for various Reynolds numbers of the calculations for waves that are normal to the flow ($k_z = 0$) plotted using the non-dimensional frequency $\omega\delta^*/U_\infty$ (δ^* = displacement thickness). The imaginary part is only of the order one-tenth of the real part showing that the waves are lightly damped as required in the theory. (From the point of view of hydrodynamic stability theory, however, these waves having a $c_i \cong -0{\cdot}08$ would be considered highly stable.) Because of their small magnitude, the imaginary part shows the highest sensitivity to the choice of integration parameters, and to small variations in the shape of the velocity profile near the wall. The vertical bar in figure 2*b* indicates the estimated uncertainty in the results (the major contribution being the uncertainty in the velocity distribution). Both the real and imaginary parts are seen to increase slowly with the Reynolds number, and the $R = \infty$ limit does not seem to have been reached at $R = 40{,}000$. It is also remarkable that the variation of α/ω is so small over the large frequency range covered, i.e. that the waves are almost non-dispersive.

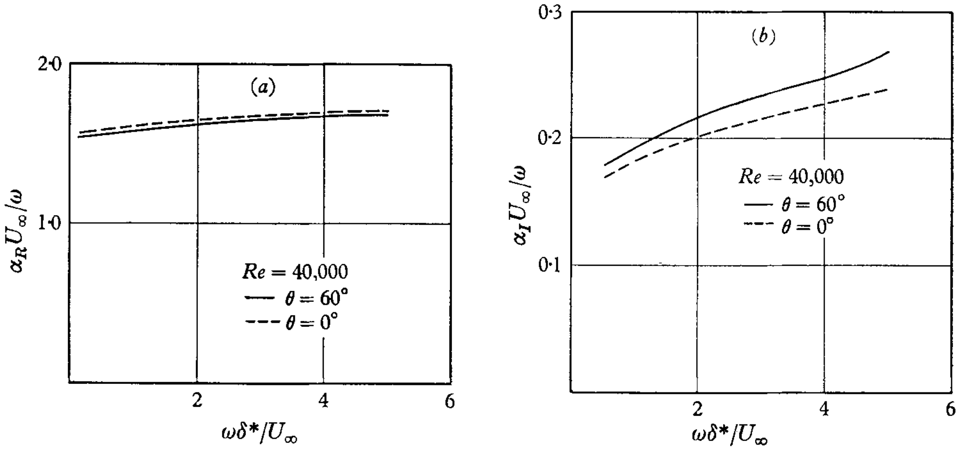

FIGURE 3. Progagation constants for oblique waves, $\theta = 60°$, at $R = 40{,}000$. (*a*) Real part. (*b*) Imaginary part.

The influence of wave obliquity ($k_z \neq 0$) is illustrated in figure 3. Here eigenvalues calculated for $R = 40{,}000$ are shown for a wave forming an angle of $\theta = 60°$ to the stream ($k_z = \alpha_R\sqrt{3}$). The real part is seen to be practically uninfluenced by wave angle, whereas the imaginary part is increased somewhat indicating a slightly higher damping. That the wave angle should have a small effect can also be shown to follow from the application of Squire's theorem to non-dispersive waves with slow variation of propagation speed and damping with the Reynolds number.

5. Results for the cross-spectral density

The calculated eigenvalues can now be inserted into the expression (29) to obtain the cross-spectral density. However, in order to evaluate the integral one also needs the function $\Phi^{(0,0)}$, which is related to the statistics of the non-linear stress terms and cannot be obtained with the aid of the present theory. Fortunately, the very small sensitivity of the eigenvalues to the wave angle, and hence to k_z, allows us to carry the results considerably further with only minor additional approximation. In the expression (29) the main variation of the integrand with k_z must clearly be due to the factor $\Phi^{(0,0)}$, and the variation of the exponential containing $\alpha^{(0)}$ can safely be ignored, since its effect on the value of the integral will be very small. We may therefore set (omitting the superscripts)

$$S_{pp}(\xi,\zeta,\omega) \simeq \frac{1}{2\pi}\exp(i\alpha_R\xi - \alpha_I|\xi|)\int_{-\infty}^{\infty}\frac{\Phi}{2\alpha_I}\exp(ik_z\zeta)\,dk_z$$
$$\equiv B(\zeta,\omega)\,S(\omega)\exp(i\alpha_R\xi-\alpha_I|\xi|),\quad \text{say,} \tag{32}$$

where

$$B = \frac{1}{2\pi S(\omega)}\int_{-\infty}^{\infty} \frac{\Phi}{2\alpha_I} \exp(ik_z\zeta)\,dk_z \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{B}(k_z)\exp(ik_z\zeta)\,dk_z.$$

Here $S(\omega)$ is the power spectrum and $B(\zeta, \omega)$ a function describing the variation of the cross-spectral density with spanwise separation, normalized in such a way that $B(0, \omega) = 1$. If we assume that the Fourier transform $\hat{B}$ is a function of the

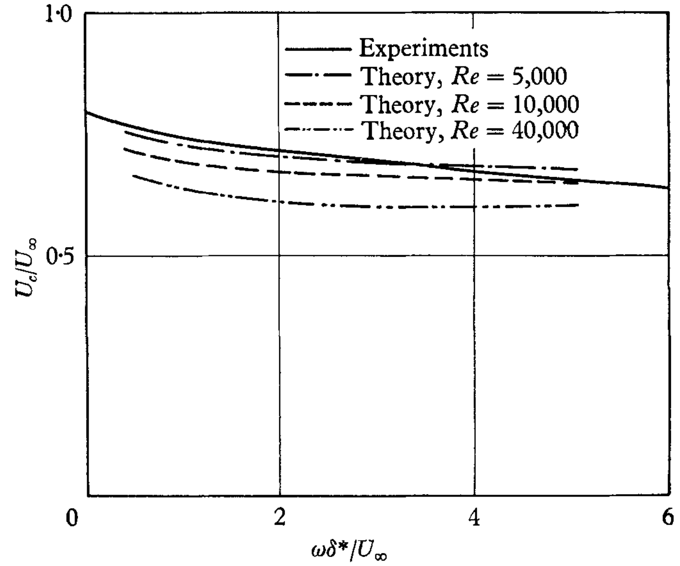

FIGURE 4. Convection velocities. Experiments by Willmarth & Wooldridge (1962) (see Corcos 1964).

wave angle, only, i.e. of k_z/α_R, rather than of k_z separately, we find that B should be a function of the variable $\alpha_R\zeta$. Defining

$$U_c \equiv \frac{\omega}{\alpha_R} = \left\{\frac{1}{\omega\xi}\tan^{-1}\left[\frac{(S_{pp})_I}{(S_{pp})_R}\right]\right\}^{-1}, \tag{33}$$

we may write the result in the following form:

$$S_{pp}(\xi, \zeta, \omega) = S(\omega)\exp\left(\frac{i\omega\xi}{U_c}\right) A\left(\frac{\omega\xi}{U_c}\right) B\left(\frac{\omega\zeta}{U_c}\right), \tag{34}$$

where
$$A = \exp\left(-d\frac{\omega\,|\xi|}{U_c}\right) \equiv \frac{|S_{pp}(\xi, \zeta, \omega)|}{|S_{pp}(0, \zeta, \omega)|}, \tag{35}$$

and
$$d = \alpha_I/\alpha_R.$$

Equation (34) is recognized as the similarity expression proposed by Corcos (1964) on basis of the experiments by Willmarth & Wooldridge (1962) using the same definition of the convection velocity U_c and the streamwise decay function A as equations (33) and (35), respectively.

The approximation (32) suggests that one should use for the eigenvalues those calculated for the value of k_z for which the transform $\hat{B}$ has the maximum.

Examination of Corcos's (1964) results for B indicates that its transform should be highly peaked around $k_z = 0$ so that the major contribution to the pressure cross-spectral density seems to come from waves that are nearly normal to the stream. Thus, the results for zero wave angle presented in figure 2 would be the most appropriate ones to use.

The convection velocity is shown in figure 4 and the exponential decay factor d in figure 5. It is seen that both are functions that vary slowly with frequency. The convection velocity decreases slowly with increasing Reynolds number, whereas the decay rate is essentially independent of R so that it can, for all practical purposes, be represented by a single curve.

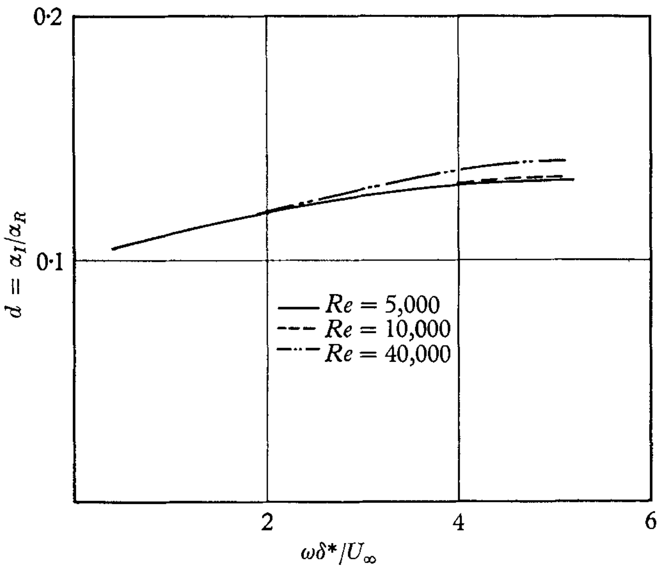

FIGURE 5. Theoretical decay coefficients.

6. Discussion of the results

Experimental values of the convection velocity as determined by Corcos (1964) from the measurements by Willmarth & Wooldridge (1962) are included in figure 4. It is seen that the theory gives convection velocities that are too low for the very high value of the laboratory Reynolds number ($R \simeq 4 \cdot 10^5$). Although consistent numerical results could not be directly obtained for such a high Reynolds number, logarithmic extrapolation indicates that the theory would give values of around $0 \cdot 5 U_\infty$ compared to the experimental ones of about $U_c = 0 \cdot 7 U_\infty$. For a Reynolds number of 5000, however, which is about 80 times smaller than the experimental value, both the magnitude of U_c and its variation with frequency is in excellent agreement with the measurements. A very significant conclusion that can be drawn from the experiments is, as shown by Corcos (1964), that the convection velocity as defined by (33) is independent of the streamwise separation distance ξ, to within the accuracy of the experiments (see his figure 7). This result strongly supports the hypothesis made in our theory that there is only one dominant wave mode present.

A comparison of the experimentally determined streamwise decay (Corcos 1964) with the theoretical results evaluated for $\omega\delta^*/U_\infty = 0{\cdot}5$ and 4 is shown in figure 6. The experiments produce very nearly the same curve for all frequencies. Willmarth & Roos (1965) suggested that the simplest analytical curve that would fit the data is an exponential of the form (35) with $d = 0{\cdot}1145$. This value is bracketed nicely by the theoretical values 0·105 and 0·130 used in the figure.

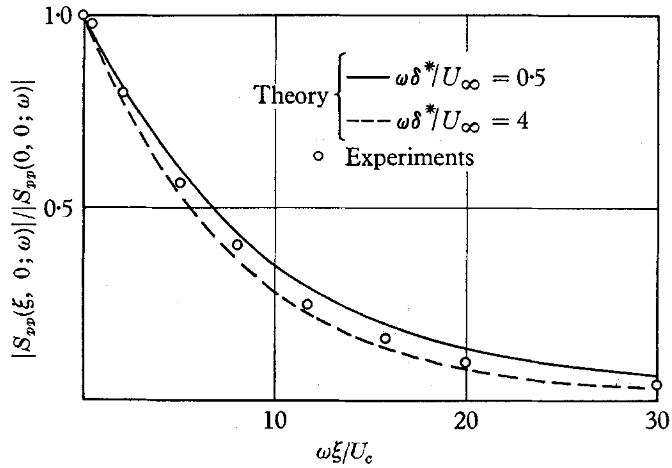

FIGURE 6. Streamwise decay of cross-spectral density. Experiments by Willmarth & Wooldridge (1962) (see Corcos 1964).

In fact, to use a constant value of d would not be inconsistent with the numerical uncertainty of the calculated decay rate (cf. figure 2). A constant decay rate implies that each eddy decays on the average in a distance proportional to its own size. Such a result is consistent with Millikan's (1938) interpretation of the logarithmic portion of the velocity profile as one for which there is no distinct length scale. Hence, the only scale that the decay of an eddy can be measured in is the size of the eddy itself.

Another interesting result of the present theory is that the similarity hypothesis proposed by Corcos (1964)—that the cross-spectral density can be written as one function of the streamwise separation distance times another function of the spanwise separation distance—follows from the insensitivity of the wave propagation constant to wave angle. Willmarth & Roos (1965) state that the experiments indicate that the similarity fails in certain cases, but the experimental results available so far are not comprehensive enough to allow a complete assessment of its validity.

A possible explanation of the curious result that the theory seems to give excellent results if a viscosity much higher than the actual molecular value is used may be arrived at from examination of the non-linear fluctuating stress term on the right-hand side of (20). In the inversion of the Fourier transform it was implicitly assumed that the Fourier transform $\hat{q}$ of q was a smooth function without poles in the neighbourhood of the real axis. The 'source term' q is a quadratic functional of u, v, w. Hence, its Fourier transform is given by a triple convolution integral. If the frequency wave-number spectrum of **u**, say, has a contribution from a wave-like disturbance of the form $C/(k_x - \alpha)$, then the Fourier transform of u^2 will be of the form

$$\hat{u^2} \sim \iiint \frac{C(\mathbf{k}-\mathbf{k}', \omega-\omega')\, C(\mathbf{k}', \omega')\, d\mathbf{k}'\, d\omega'}{[k_x - \alpha(k_z', \omega')]\,[k_x - k_x' - \alpha(k_z - k_z', \omega - \omega_s)]}.$$

The integration over k_x' can be carried out under the assumption that the poles are near the real axis. One finds that

$$\hat{u^2} \sim \iint \frac{C(\alpha, k_z', \omega')\, C(k_x - \alpha, k_z - k_z', \omega - \omega')\, dk_z' d\omega'}{k_x - \alpha(k_z', \omega') - \alpha(k_z - k_z', \omega - \omega')}.$$

For a non-dispersive wave one has that α is proportional to ω, $\alpha = K\omega$, say, and α is independent of k_z if the Reynolds number dependence is assumed negligible. Then

$$\hat{u^2} \sim \iint \frac{C(\alpha, k_z', \omega')\, C(k_x - \alpha, k_z - k_z', \omega - \omega')\, dk_z' d\omega'}{k_x - K\omega}$$

$$= \frac{1}{k_x - \alpha} \iint C(\alpha, k_z', \omega')\, C(k_x - \alpha, k_z - k_z', \omega - \omega')\, dk_z' d\omega'.$$

Hence $\hat{u^2}$ will have a pole at $k_x = \alpha$, and the transform of q will thus have a pole of order two. The wave-number frequency spectrum will have terms like $|k_x - \alpha|^{-4}$ which upon inversion will give rise to terms of the form $\xi^2 \exp(i\alpha_R \xi - \alpha_I |\xi|)$ in the cross-spectral density. Clearly, such 'secular' behaviour is a consequence of the divergence of the formal iterative solution at $k_x = \alpha$ in the non-dispersive case. To remove the singularity, one could add a small linear term on both sides of the equation such that

$$\iint C(\alpha, k_z', \omega')\, C(k_x - \alpha, k_z - k_z', \omega - \omega')\, dk_z' d\omega' = 0$$

for all ω and k_z. An equivalent procedure is to calculate the propagation constant for a sinusoidal (in time) perturbation of infinitesimal amplitude superimposed on the pre-existing fluctuating flow field. If the fluctuations are known, this can in principle be done by expansion in the amplitude of the fluctuations. This will require going to terms that are of the order square of the fluctuating velocities times the amplitude of the wave (only terms linear in the infinitesimal amplitude of the wave need of course be retained). Such an investigation would be exceedingly complicated. However, it seems likely on physical grounds that the scattering effect on the wave due to the pre-existing fluctuations would be largely a dissipative one perhaps having the overall effect of an eddy viscosity.

7. Conclusions

The basic aim of the present paper was to bring out the importance of the wave propagation mechanism due to the shear flow on the statistical properties of the turbulent fluctuations. It was shown that whenever lightly damped waves (of random phase and orientation) were present, they tended to dominate over the other kinds of disturbances, the reason being that such waves can correlate over a large distance of the order of the inverse of their damping ratio.

In the analysis, the equations of motion for the fluctuations in a parallel shear flow were manipulated to produce a non-homogeneous Orr–Sommerfeld equation with the non-linear fluctuating turbulent stress terms on the right-hand side of this equation being considered as a known forcing term. The possibility of wave propagation was then brought out through an expansion in terms of the eigenfunctions of the homogeneous Orr–Sommerfeld problem. Numerical values for the least damped eigenvalue for this problem, and hence for the dominating wave propagation modes, were obtained by the aid of a computer. The results were then used to calculate approximately the convection speed and stream-wise decay of the fluctuations as functions of the non-dimensional frequency, quantities that could be directly compared with experimental values. For the stream-wise decay, excellent qualitative and quantitative agreement was found. Thus, each wave component was found to decay by a factor of e^{-1} in a distance of about 1·4 times its wavelength and to have practically lost its identity completely after a distance of 6 times its length in agreement with the measurements by Willmarth & Wooldridge (1962). That the only appropriate length scale in which the development of an individual eddy can be measured is the size of the eddy itself is a result in complete consistency with Millikan's (1938) interpretation of the logarithmic profile as one in which there is no specific length scale. The convection velocities given by the theory, on the other hand, turned out to be too low by about 30 %. Excellent agreement was obtained, however, for a value of the viscosity about 80 times the experimental value. The conclusion drawn is that for a more accurate calculation of convection velocities one needs to consider the propagation of an infinitesimal wave in a flow with pre-existing fluctuations. No attempts have been made to attack this very difficult problem, but one may speculate that the fluctuations will act qualitatively like an eddy viscosity, at least for the long waves. Preliminary calculations with the eddy viscosity included in the linear perturbation equation in the manner proposed by Betchov & Criminale (1964) give a very much improved agreement for the convection velocity at lower frequencies ($\omega\delta_*/U_\infty < 1{\cdot}5$), but at the higher frequencies the predicted values were again too low. Of course, the use of a quasi-steady eddy viscosity can only be justified for waves of frequencies that are very much lower than those of the majority of the Reynolds stress-producing eddies. Somewhat surprisingly, viscosity was found to have considerably less influence on the decay rate $d = \alpha_I/\alpha_R$ than on the convection velocity. This may be a partial explanation of the success for the former quantity, despite a possibly incorrect 'effective viscosity' employed. The theory was found to confirm Corcos's (1964) similarity hypothesis which could be shown to be a consequence of the insensitivity of the wave-propagation constant to wave orientation angle.

The present results apparently contradict some of the basic hypotheses made by Malkus (1956) in his interesting theory of shear-flow turbulence. He postulated that the smallest scale of turbulence would be determined from the condition of marginal stability. The present theory shows that, for a shear flow that is statistically homogeneous in the direction of the flow (as is indeed the case for the problem considered by Malkus; the flow between parallel plates), all waves must be damped, otherwise disturbances from the leading edge or entrance of the flow would propagate any distance downstream and hence destroy the homogeneity. The numerical eigenvalue calculations tend to confirm this; no unstable or marginally stable eigenvalue in the Orr–Sommerfeld sense was found despite extensive variations of the parameters involved. In fact, all waves turned out to have about the same damping ratio and are highly stable from the point of view of laminar stability theory. A recent calculation by Reynolds & Tiederman (1967) employing the asymptotic method also confirms the present finding that the mean velocity profile is stable. These authors were able to show that Malkus's apparent success in predicting the constants involved in the logarithmic profile might have been due to an inaccurate estimate of the stability boundary. However, it should be pointed out that the velocity profile considered by Malkus was different from the one used here and by Reynolds & Tiederman (1967). His profile had a second derivative obtained as the square of a finite Fourier series which could have a finite number of zeros and therefore be susceptible to inflexional instability. In contrast, a mean velocity profile given by an expression like (31) always has a very large negative second derivative and would therefore be expected to be highly stable. It would seem that Malkus's ideas on the stability might have more relevance for the *instantaneous* velocity profile which undoubtedly will have local regions of strong inflexional instability in the manner considered by Greenspan & Benney (1963) in their analysis of turbulent bursts in the final stage of transition.

Malkus (1956) also hypothesized that the non-linear terms have a stabilizing influence on the fluctuations. In the present model, the non-linear terms are actually considered to be the driving terms, and hence, in a sense, de-stabilizing. That non-linear terms can cause instability is known from the phenomenon of turbulent bursts (Greenspan & Benney 1963). The present analysis does not attempt to shed any light on the non-linear driving mechanism itself. What is envisaged in the model is rather the long-range effect of a local break-down being propagated in the form of waves. Recent experimental research by Runstadler, Kline & Reynolds (1963) and Tu & Willmarth (1966), gives strong indications that an essential part of this mechanism is provided by highly swept eddies shooting off in a random fashion from the laminar wall layer into the turbulent portion. It is clear that such a process will cause direct non-linear interaction between wave-numbers over a large range, and that the traditional conceptual picture of large turbulent eddies breaking down into smaller ones, and these in turn into even smaller ones until viscosity takes over is unlikely to be of much help. It is also doubtful whether one could attack the non-linear sustaining mechanism by conventional statistical methods. For example, the cross-spectral density was shown to be dominated entirely by the least damped wave, but these may or may not by themselves be important in the non-linear interaction process.

A fortunate circumstance was that the present analysis could be carried through with a minimum of assumptions. The restriction to an incompressible flow is not essential and can in principle be relaxed. The stability theory for a compressible parallel flow is well developed (Mack 1965) but the numerical calculations are considerably more complicated and time-consuming. As to the assumption of a parallel flow it is known to be an excellent one, even for shear flows which vary much more rapidly in the stream direction than the flat-plate boundary layer. However, if an individual wave persists for a distance over which the shear profile or shear layer thickness changes considerably, one must account for this variation. In such cases, streamwise homogeneity is of course destroyed. Such will be specifically the case for a jet or a free shear layer or other shear layers with inflexion points. For these, there would be frequency ranges for which the wave will be unstable. A wave of a given frequency would not grow indefinitely large, however, because due to the growth of the shear layer, it will eventually travel into a stable region. Whether the basic idea of the dominance of the linear wave modes would hold also for unstable flows is not certain, however, because of the very large fluctuations known to be present in such flows.

Many possible extensions of the present theory are obvious. Thus, for example, the validity of the assumption that the higher eigenvalues are considerably more damped and that therefore only the least damped wave mode needs to be considered, can be directly verified by the calculation of the next eigenvalue. The computational program used did not allow a systematic exploration of the higher eigenvalues, but occasional results that were obtained by chance indicate that the eigenvalue for the next mode has a damping ratio of about twice the lowest one.

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